Maximal clusters in non-critical percolation and related models

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Abstract: We investigate the maximal non-critical cluster in a big box in various percolation-type models. We investigate its typical size, and the fluctuations around this typical size. The limit law of these fluctuations are related to maxima of *independent* random variable with law described by a single cluster.

Key-words: Maximal clusters, exponential law, Gumbel distribution, FKG inequality, second moment estimates.

1 Introduction and main results

Bazant in [6] studies the distribution of maximal subcritical clusters, both numerically and via a non-rigorous renormalization group argument. He finds that the cardinality of maximal clusters behaves like the maximum of independent geometrically distributed random variables, i.e., a "Gumbel-like" distribution. In his paper, the role of the FKG inequality, which means that clusters "repel each other", is already emphasized in a subadditivity argument.

In this paper, we rigourously prove these claims for a broad class of non-critical percolation type models. In the FKG context, we can deal both with maximal subcritical and finite supercritical clusters, and obtain a Gumbel distribution for both. In a more general context, we can deal with dependent percolation models dominated by subcritical Bernoulli percolation.

The key ingredient of the proof of the Gumbel law is to use the exponential law for the occurrence time of rare patterns. This idea is used by Wyner in [29] in the context of matching two random sequences. If a cluster bigger than u_n appears in a box $[-n,n]^d \cap \mathbb{Z}^d$ of volume $(2n+1)^d$, then evidently the occurrence time $\mathbf{t}_{u_n}$ of such a cluster is less than $(2n+1)^d$. Therefore, if $\mathbf{t}_{u_n}$ has approximately an exponential distribution, then the probability of having a cluster larger than u_n is approximately $1-e^{-(2n+1)^d \mathbb{P}(\mathcal{C}_{u_n})}$, where $\mathcal{C}_{u_n}$ denotes the event that the cluster of the origin has cardinality at least u_n. If one can find a scale $u_n = u_n(x)$ such that $\mathbb{P}(\mathcal{C}_{u_n(x)}) \simeq e^{-x}/(2n+1)^d$, then one obtains the Gumbel law. Assuming an exponential decay of the cluster cardinality, as expected for subcritical percolation, one obtains $u_n(x) = u_n + x$, where $u_n = c\log n + o(\log n)$. For finite supercritical clusters, under the assumption of Weibull-tails where the tails decay as a stretch exponential with exponent $\delta < 1$, we have $u_n(x) = (c\log n + c' \log\log n + x)^{1/\delta}$.

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1.1 The model

We consider site percolation and related models on the lattice $\mathbb{Z}^d$. A configuration of occupied and vacant sites is an element $\omega \in \Omega = \{0,1\}^{\mathbb{Z}^d}$. A site x with $\omega(x) = 1$ is called occupied, and a site with $\omega(x) = 0$ is called vacant.

The configuration ω will be distributed according to a translation invariant probability measure $\mathbb{P}$ on the Borel-σ-field of Ω. Examples of $\mathbb{P}$ include the Bernoulli product measure $\mathbb{P}_p$ with $\mathbb{P}_p(\omega(x) = 1) = p$, but we will also consider dependent random fields, such as the Ising model, below.

A set $A \subseteq \mathbb{Z}^d$ is *connected* if for any $x, y \in A$ there is a nearest-neighbor path γ joining x and y. The cluster $\mathcal{C}(x) = \mathcal{C}(x,\omega)$ of an occupied site x is the largest connected subset of occupied sites to which x belongs. By convention, $\mathcal{C}(x) = \varnothing$ if $\omega(x) = 0$. We will also need the cluster $\mathcal{C}_{\mathrm{le}}(x)$ defined as follows

$$\mathcal{C}_{\mathrm{le}}(x) = \begin{cases} \mathcal{C}(x) & \text{if } x \text{ is the left endpoint of } \mathcal{C}(x), \\ \varnothing & \text{otherwise.} \end{cases} \tag{1.1}$$

Here by the left-endpoint of a finite set $A \subseteq \mathbb{Z}^d$, we mean the minimum of A in the lexicographic order. By definition $\mathcal{C}_{\mathrm{le}}(x) \cap \mathcal{C}_{\mathrm{le}}(y) = \varnothing$ if $x \neq y$. In this paper, we will work with site percolation. In the percolation community, it is more usual to consider bond percolation (see e.g. [23]). However, site percolation is more general than bond percolation, as shown e.g. in [23, Section 1.6]. We will use results from [23] proved for bond percolation, but in general these results also hold for site percolation (as noted in [23, Section 12.1]).

Percolation has a *phase transition*, i.e., for $d \geq 2$, there exists a critical value $p_c \in (0,1)$ such that there exists an infinite cluster a.s. for $p > p_c$, while no such cluster exists a.s. for $p < p_c$. The goal of this paper is to investigate maximal clusters in a finite box for $p \neq p_c$.

1.2 Main results for site percolation

In this section, we describe our results in the simplest case, namely for site percolation, where all vertices are independently occupied with probability p and vacant with probability $1 - p$.

We will study the maximal cluster inside a big box. To be able to state our result, we need some further notation. Let $B_n = [-n,n]^d \cap \mathbb{Z}^d$ be the cube of width $2n+1$. We let

$$\omega_{B_n} = \begin{cases} \omega(x) & \text{if } x \in B_n, \\ 0 & \text{otherwise,} \end{cases} \tag{1.2}$$

and

$$\mathcal{M}_n = \mathcal{M}_n(\omega) = \max_x |\mathcal{C}_{\mathrm{le}}(x, \omega_{B_n})|. \tag{1.3}$$

The random variable $\mathcal{M}_n$ is the maximal cluster inside B_n, with zero boundary conditions, i.e., where we do not consider connections outside B_n. The goal of this paper is to obtain an extreme value theorem such as

$$\mathbb{P}\left(\mathcal{M}_n \leq u_n + x\right) = e^{-a_n e^{-x}} + o(1) \tag{1.4}$$

for some $u_n \uparrow \infty$, and where a_n is a bounded sequence. In words, this means that the distribution of the maximal cluster is "Gumbel-like", i.e., looks like the maximum of independent geometric random variables. The presence of the bounded sequence a_n is typical for the law of the maximum of independent geometric random variables, where we do not have an exact limiting extreme value distribution cannot (see e.g. [20, Corollary 2.4.1]).

The idea developed in this paper is that for any non-critical p, the law of $\mathcal{M}_n$ is asymptotically equal to the law of the maximum of $(2n+1)^d$ *independent* copies of a random variable X with law

$$\mathbb{P}(X = n) = \frac{1}{n}\mathbb{P}(|\mathcal{C}(0)| = n), \tag{1.5}$$

for $n \geq 1$, and

$$\mathbb{P}(X = 0) = 1 - \mathbb{E}(|\mathcal{C}(0)|^{-1}). \tag{1.6}$$

The law of X in (1.5–1.6) turns out to be equal to the law of the random variable $|\mathcal{C}_{\mathrm{le}}(0)|$ (see Lemma 4.1 below). Therefore, the law of $\mathcal{M}_n$ is equal to the law of $\max_x |\mathcal{C}(x, \omega_{B_n})|$, and thus the philosophy of the paper is to show that the clusters are only weakly dependent. We further use properties of the law of $\mathcal{C}(0)$ to derive the asymptotics of $\mathcal{M}_n$ in more detail.

We note that the cluster size distribution will play an essential part throughout the proof. We will now state the results on this cluster size distribution which we need, in order to specialize the results. Since this law is crucially different for $p < p_c$ and $p > p_c$, we distinguish these two cases.

For $p < p_c$, it is shown in [23, Theorem (6.78)] that

$$\zeta(p,d) = \lim_{n\to\infty} -\frac{1}{n}\log \mathbb{P}_p(|\mathcal{C}(0)| \geq n) \tag{1.7}$$

exists, and that $\zeta = \zeta(p,d) > 0$ for all $p < p_c$. Moreover, there exists $C = C(p)$ such that

$$\mathbb{P}_p(|\mathcal{C}(0)| = n) \leq Cne^{-\zeta n}. \tag{1.8}$$

We will sometimes work under an assumption that a somewhat stronger version of (1.7) holds, namely that

$$\lim_{n\to\infty} \frac{\mathbb{P}_p(|\mathcal{C}(0)| \geq n+1)}{\mathbb{P}_p(|\mathcal{C}(0)| \geq n)} = e^{-\zeta}. \tag{1.9}$$

Assumption (1.9) is stronger than (1.7), but strictly weaker than the widely believed tail-behavior, namely that there exist $\theta = \theta(d) \in \mathbb{R}$ and $A = A(p,d)$ such that

$$\mathbb{P}_p(|\mathcal{C}(0)| \geq n) = An^{\theta}e^{-\zeta n}[1 + o(1)]. \tag{1.10}$$

Our main result for $p < p_c$ is the following theorem:

Theorem 1.1. *Fix $p < p_c$ and assume that (1.9) holds. Then there exists a sequence $u_n \in \mathbb{N}$, with $u_n \to \infty$, a real number $a > 0$ and a bounded sequence $a_n \in [a, 1]$, such that for all $x \in \mathbb{N}$*

$$\mathbb{P}(\mathcal{M}_n \leq u_n + x) = e^{-a_n e^{-x\zeta}} + o(1). \tag{1.11}$$

Theorem 1.1 shows that $\mathcal{M}_n$ is bounded above and below by Gumbel laws, and shows in particular that the sequence $\mathcal{M}_n - u_n$ is tight. Our proof will reveal that Theorem 1.1 can be extended to yield weak convergence along certain exponentially growing sequences.

We now go to supercritical results. Since $p_c(1) = 1$, we may assume that we are in dimension $d > 1$. When $p > p_c$, then it is shown in [23, Theorem (8.61) and (8.65)] that there exist $\eta = \eta(p,d)$ and $\gamma = \gamma(p,d)$ such that

$$e^{-\gamma n^{\frac{d-1}{d}}} \leq \mathbb{P}_p(|\mathcal{C}(0)| = n) \leq e^{-\eta n^{\frac{d-1}{d}}}. \tag{1.12}$$

In $d = 2, 3$, it is known that the limit

$$\eta(p, d) = \lim_{n\to\infty} -\frac{1}{n^{\frac{d-1}{d}}} \log \mathbb{P}_p(n \leq |\mathcal{C}(0)| < \infty). \tag{1.13}$$

exists. The limit in (1.13) is related to the large deviations of large finite supercritical clusters, and can be written explicitly as a variational problem over possible cluster shapes. This variational problem involves the surface tension, and is maximized by the so-called Wulff shape. The result in $d = 2$ is in [4, 14], while for $d = 3$, it is in [15].

We will again formulate a stronger version of (1.13), namely that for every $x \in \mathbb{R}$, we have

$$\lim_{n\to\infty} \frac{\mathbb{P}_p(n + xn^{1/d} \leq |\mathcal{C}(0)| < \infty)}{\mathbb{P}_p(n \leq |\mathcal{C}(0)| < \infty)} = e^{-x\eta\frac{d-1}{d}}, \tag{1.14}$$

and change the definition of $\mathcal{M}_n$ slightly to

$$\mathcal{M}_n = \mathcal{M}_n(\omega) = \max_{x:|\mathcal{C}_{\mathrm{le}}(x)|<\infty} |\mathcal{C}_{\mathrm{le}}(x, \omega_{B_n})|, \tag{1.15}$$

i.e., we take the largest *finite* cluster. Of course, for $p < p_c$ (1.3) and (1.15) coincide.

Then we can prove the following scaling property:

Theorem 1.2. *Fix $p > p_c$ and assume that (1.14) holds. Then there exists a sequence $u_n \to \infty$ with such that for all $x \in \mathbb{R}$*

$$\lim_{n\to\infty} \mathbb{P}(\mathcal{M}_n \leq u_n + xu_n^{1/d}) = e^{-e^{-x\eta\frac{d-1}{d}}}. \tag{1.16}$$

Theorems 1.1 and 1.2 study fluctuations of $\mathcal{M}_n$ around their asymptotic mean under the Assumptions (1.9) and (1.14). The main difference between Theorems 1.1 and 1.2 is that Theorem 1.2 implies weak convergence of the rescaled $\mathcal{M}_n$ since the fluctuations grow with n, whereas in Theorem 1.1 this weak convergence does not hold due to the fact that the fluctuations are of order 1 so that the discrete nature of cluster sizes persists.

In Section 3 below, we will formulate more general results that hold *without* Assumptions (1.9) and (1.14), but that take a form which is less elegant. It is not so hard to see that one can choose

$$u_n = O(\log n) \tag{1.17}$$

for $p < p_c$, while

$$u_n = O((\log n)^{\frac{d}{d-1}}) \tag{1.18}$$

when $p > p_c$. From Theorems 1.1 and 1.2 it immediately follows that $\mathcal{M}_n$ divided by $\log n$ for $p < p_c$, respectively, $(\log n)^{\frac{d}{d-1}}$ for $p > p_c$, converges in probability to a constant. In the next theorems, we will investigate the typical size of $\mathcal{M}_n$ in more detail and prove convergence almost surely.

Theorem 1.3. *For $p < p_c$,*

$$\frac{\mathcal{M}_n}{\log n} \to d\zeta(p, d) \qquad a.s. \tag{1.19}$$

Theorem 1.4. *For $p > p_c$, and $d = 2, 3$,*

$$\frac{\mathcal{M}_n}{(\log n)^{\frac{d}{d-1}}} \to d^{\frac{d-1}{d}}\eta(p, d) \qquad a.s. \tag{1.20}$$

For $d \geq 4$, if the limit in (1.13) exists, then (1.20) holds.

We close this section with a few observations concerning the role of the boundary conditions. In (1.3), we have taken the maximal cluster under the zero boundary condition, so that we can write $\mathcal{M}_n = \mathcal{M}_n^{(\mathrm{zb})}$. Alternatively, we could defined $\mathcal{M}_n$ under free boundary conditions, i.e.,

$$\mathcal{M}_n^{(\mathrm{fb})} = \max_{x \in B_n} |\mathcal{C}(x, \omega)|, \tag{1.21}$$

or under periodic boundary conditions, i.e.,

$$\mathcal{M}_n^{(\mathrm{pb})} = \max_{x \in B_n} |\mathcal{C}(x, \omega'_{B_n})|, \tag{1.22}$$

where ω'_{B_n} is the site percolation configuration on the torus with vertex set B_n. We will finally show that this makes no difference whatsoever:

Theorem 1.5. *For $p < p_c$,*

$$\mathbb{P}_p(\mathcal{M}_n^{(\mathrm{zb})} \neq \mathcal{M}_n^{(\mathrm{fb})}) = o(1), \qquad \mathbb{P}_p(\mathcal{M}_n^{(\mathrm{zb})} \neq \mathcal{M}_n^{(\mathrm{pb})}) = o(1). \tag{1.23}$$

For $p > p_c$,

$$\mathbb{P}_p(\mathcal{M}_n^{(\mathrm{zb})} \neq \mathcal{M}_n^{(\mathrm{fb})}) = o(1). \tag{1.24}$$

Theorem 1.5 immediately shows that all results proved for $\mathcal{M}_n^{(\mathrm{zb})}$ immediately also apply to $\mathcal{M}_n^{(\mathrm{fb})}$ and $\mathcal{M}_n^{(\mathrm{pb})}$ for $p < p_c$ and to $\mathcal{M}_n^{(\mathrm{fb})}$ for $p > p_c$i.e., that the boundary condition is irrelevant. For $p > p_c$, $\mathcal{M}_n^{(\mathrm{pb})}$ is more difficult to work with since it is harder to 'exclude' the infinite cluster on the torus without looking outside the torus.

1.3 Discussion of the results

In this section, we discuss our results and their relation to the literature.

1.3.1 Runs and one-dimensional site percolation

In the case where $d = 1$, it easily follows that for any $p < p_c = 1$,

$$\mathbb{P}(|\mathcal{C}_{\mathrm{le}}(0)| \geq n) = p^n. \tag{1.25}$$

In this simple case, the largest cluster is equal to the longest run of ones in n independent tosses. This is a classical problem, and the leading order asymptotics $\mathcal{M}_n = \log n / \log p(1 + o(1))$ is the celebrated Erdös-Rényi law [19]. Our results studies fluctuations around the Erdös-Rényi law. This problem has attracted considerable attention due to its relation to matching problems arising in sequence alignment (see e.g., [27] and the references therein).

1.3.2 Results for general subcritical FKG models and related Gumbel laws

Our results for subcritical clusters hold more generally than just for independent site percolation. The main technical ingredient in the proof are the FKG-inequality and bounds on the tails of the cluster size distribution. In Section 3 below, we will state a general result, that can be proved for site percolation and applied in the context of the following examples.

1. The two-dimensional Ising model at $\beta < \beta_c$.

2. The Ising model in general dimension, at high temperature and/or high enough magnetic field (see [22]).

3. Gibbs measures where the potential has a sufficiently small Dobrushin norm and a sufficiently high magnetic field.

See [22] for an introduction of the Ising model and Section 3 for more details.

We expect that related results hold for other maximal values of cluster characteristics. Examples are the maximal diameter of a supercritical finite cluster, or the maximal occupied line (i.e., a sequence of bonds) with any orientation for $p < 1$. We also expect that our results for maximal finite supercritical clusters continue to hold in the context of the Ising model in dimensions $d = 2, 3$ for $\beta > \beta_c$, where the Wulff crystal has been identified (see e.g. [7, 16]), and hence the exact behavior of the cluster tail is known. We use the version of the exponential law from [2], and this does not hold for the low-temperature Ising model. The weaker version of the exponential law proved in [17] does apply, but it is unclear whether we can apply this result in the present setting. The proofs of Theorems 1.3–1.4 are more robust, and are likely to apply to the Ising model as well.

1.3.3 Maximal clusters for critical percolation

Our results are only valid for non-critical percolation. In critical percolation, the behavior of the largest cluster in a box should be entirely different. Firstly, the scaling of the largest cluster in a box should be *polynomial* in the volume of the box, rather then *polylogarithmic* as in Corollaries 1.3 and 1.4. Secondly, when properly rescaled, the size of the largest cluster should converge to a proper random variable, rather than to a constant as in Corollaries 1.3 and 1.4. Thirdly, we expect that in some cases, the size of the largest cluster depends on the boundary conditions, which is not true off the critical point (see Theorem 1.5).

There have been results in the direction of the above claims. In [13], the largest critical cluster in a box was investigated and, under certain scaling assumptions, it was proved that the largest cluster with zero boundary conditions scales like $n^{\frac{d\delta}{\delta+1}}$, where δ is the critical exponent related to the critical cluster distribution

$$\frac{1}{\delta} = \lim_{n\to\infty} -\frac{\log \mathbb{P}_{p_c}(|\mathcal{C}(0)| \geq n)}{\log n}. \tag{1.26}$$

Of course, it is not obvious that this limit exists. The scaling assumptions are not expected to be true above the critical dimension $d_c = 6$. In [9], it is conjectured that the same should be true for the largest cluster with periodic boundary conditions, i.e., for percolation on the torus.

Above the critical dimension $d_c = 6$, other scaling occurs. Aizenman [3] proves that, under a certain assumption on the two-point function, that the largest cluster has size n^4, and that there are n^{d-6} clusters of such order. The assumption was proved to hold for nearest-neighbor bond percolation in sufficiently high dimension in [24], and for sufficiently spread-out percolation above 6 dimensions in [25]. For periodic boundary conditions, Aizenman [3] conjectured that the scaling should be like $n^{2d/3}$. Partial results in this direction have appeared in [9, 10]. It is a well-know result that for the critical random graph, where $p = 1/N$, and N the size of the graph, the largest cluster is of the order $N^{2/3}$. Thus, Aizenman's conjecture amounts to the conjecture that the largest critical cluster with periodic boundary conditions scales like the largest critical cluster for the random graph (see also Section 1.3.4 below).

1.3.4 Relation to random graphs

There is a wealth of related work for random graphs, which are finite graphs where edges are removed independently. This research topic was started by a seminal paper of Erdös and Rényi [18], which created the field of random graphs. Erdös and Rényi investigate what is called *the* random graph, i.e., the complete graph where edges are kept independently with fixed probability p and removed otherwise. See the books [5, 8, 26] and the references therein. The fields of percolation and random graphs have to a large extent evolved independently, with different terminology and methodology. Only recently, attempts have been made to use the developed methodology in the other fields (see e.g. [12, 13, 9, 10, 11]). When dealing with random graphs, it is natural to investigate the largest connected component or cluster when the size of the graph tends to infinity. Therefore, results such as the ones presented in Section 1.2 have appeared in this field. In particular, detailed estimates of large subcritical clusters and supercritical cluster have been obtained. Of course, for *finite* graphs, it is already non-trivial to define what a critical value is. Above the critical value, the largest cluster has a size of order of the size of the graph, while below the critical value, the largest cluster is logarithmic in the size of the graph.

In random graph theory, often there is a *discrete duality principle*, which means that when we remove the largest supercritical cluster, then the size and distribution of the remaining clusters is very much alike the size and distribution of subcritical clusters. See e.g. [5, Section 10.4] for an explanation of this principle for branching processes as well as for the random graph. We note that this principle is *false* for site percolation, as Theorems 1.3 and 1.4 show. This distinction arises from the fact that the random graph has no *geometry*, whereas the geometry is essential in the description of large finite supercritical clusters and appears prominently in the Wulff shape. It would be of interest to apply our methods to random graphs.

1.3.5 Organization

Our paper is organized as follows. In Section 2, we give heuristics for our results. In Section 3, we state our general results for FKG models under certain conditions. Section 4 is devoted to the proofs of the main results.

2 Extremes and rare events: heuristics

We will be interested in the cardinality of maximal clusters inside a big box. Recall that $B_n = [-n,n]^d \cap \mathbb{Z}^d$. For $n \in \mathbb{N}$, define the σ-field $\mathcal{F}_n = \mathcal{F}_{B_n}$. A *pattern* A_n is a configuration with support on B_n, i.e., it is an element of $\{0,1\}^{B_n}$. We will identify a pattern with its cylinder, i.e., we will also denote A_n to be the set of those ω such that $\omega_{B_n} = A_n$. For a pattern A_n, we define its occurrence time to be

$$\mathbf{t}_{A_n}(\omega) = \min\Big\{|B_k| : \exists x \in \ B_k \text{ such that } \ B_n + x \subseteq B_k \text{ and } \theta_x\omega_{B_n}, = A_n\Big\} \tag{2.1}$$

where $\theta_x\omega$ denotes the configuration ω shifted over x, so that $(\theta_x\omega)(y) = \omega(x+y)$. In words, this is the volume of the minimal cube B_k which "contains" the pattern A_n. One expects that $\mathbf{t}_{A_n}$ is of the order $\mathbb{P}(A_n)^{-1}$. For $E_n \in \mathcal{F}_n$, there exists a unique set of patterns $\mathcal{A}(E_n)$ such that:

$$E_n = \bigcup_{A_n \in \mathcal{A}(E_n)} A_n$$

The occurrence time of E_n is then defined as

$$\mathbf{t}_{E_n}(\omega) = \min_{A_n \in \mathcal{A}(E_n)} \mathbf{t}_{A_n} \tag{2.2}$$

In words, E_n is a set of patterns, and the occurrence time of E_n is the volume of the first cube B_k in which some pattern of E_n can be found. A sequence of $\mathcal{F}_n$-measurable events E_n is called a *sequence of rare events* if $\mathbb{P}(E_n) \to 0$ as $n \to \infty$. For sequences of rare events, one typically expects so-called *exponential laws*, i.e., limit theorems of the type

$$\mathbb{P}\left(\mathbf{t}_{E_n} \geq \frac{t}{\mathbb{P}(E_n)}\right) = e^{-\lambda_{E_n} t} + o(1). \tag{2.3}$$

Equation (2.3) has been proved for "high temperature Gibbsian random fields" and the parameter λ_{E_n} is bounded away from zero and infinity. In the case of patterns, the parameter depends on the self-repetitive structure of the pattern. For so-called good (meaning that there can be no fast returns) patterns, we even have that $\lambda_{E_n} = 1$. In [2] the exponential law for patterns is generalized to measurable events $E_n \in \mathcal{F}_n$, provided a second moment condition is satisfied. This second moment condition ensures that $1/\mathbb{P}(E_n)$ is the right time scale for the occurrence time, i.e., the parameter λ_{E_n} is bounded away from zero and infinity (see Theorem 3.5 below for the precise formulation).

The relation between maxima and rare events is intuitively obvious: if a cluster with cardinality bigger than m appears in a cube B_n, then the occurrence time for the appearance of a cluster bigger than m is not larger than $|B_n|$. More precisely, define

$$E_n = \{n \leq |\mathcal{C}_{\mathrm{le}}(0)| < \infty\} \tag{2.4}$$

and define the random variable τ_{E_m} with values in $\{(2n+1)^d : n \in \mathbb{N}\}$ by

$$\{\tau_{E_m} \leq (2n+1)^d\} = \{\exists x \in B_n : \theta_x \omega_{B_n} \in E_m\} \tag{2.5}$$

The random variable τ_{E_m} is not exactly equal to the occurrence time $\mathbf{t}_{E_m}$, but we will see that asymptotically τ_{E_m} and $\mathbf{t}_{E_m}$ have the same distribution (see Lemma 4.5 below).

The advantage of working with τ_{E_m} lies in the equality

$$\{\mathcal{M}_n \geq m\} = \{\tau_{E_m} \leq (2n+1)^d\} \tag{2.6}$$

If we assume that the exponential law holds for the occurrence time, then

$$\mathbb{P}\left(\mathcal{M}_n \geq u_n(x)\right) \approx 1 - e^{-\lambda_{E_n} \mathbb{P}(E_{u_n(x)})(2n+1)^d} \tag{2.7}$$

So if we can choose $u_n(x)$ such that

$$\mathbb{P}(E_{u_n(x)}) \approx \frac{a_n e^{-x}}{(2n+1)^d} \tag{2.8}$$

then we obtain (1.4). This is the guiding idea of this paper, and the proof of a result of the type (1.4) thus relies on the following three ingredients:

1. Verification of the validity of the exponential law for the events E_n. For this, we will rely on the techniques developed in [2], which requires natural mixing conditions and a second moment estimate, see (cf. (3.9)).

2. Proof of the existence of the sequence $u_n(x)$ such that (2.8) holds.

3. Proof that $\lambda_{E_{u_n(x)}} = 1 + o(1)$.

3 General results

In this section, we introduce the conditions needed and state the precise form of (1.4). We will start by defining the main conditions in Section 3.1, we will state the exponential law proved in [2] in Section 3.2, and in Section 3.3, we will state our main results valid under the formulated conditions.

3.1 The conditions

There will be three main conditions, a non-uniformly exponentially φ-mixing condition, a finite energy condition, and a condition ensuring that clusters are subcritical or supercritical.

We first introduce the so-called "high mixing" condition which is adapted to the case of Gibbsian random fields. For $m > 0$ define

$$\varphi(m) = \sup \frac{1}{|A_1|} |\ \mathbb{P}\left(E_{A_1} | E_{A_2}\right) - \mathbb{P}\left(E_{A_1}\right)|, \tag{3.1}$$

where the supremum is taken over all finite subsets A_1, A_2 of $\mathbb{Z}^d$, with $d(A_1, A_2) \geq m$ and $E_{A_i} \in \mathcal{F}_{A_i}$, with $\mathbb{P}(E_{A_2}) > 0$. Note that this $\varphi(m)$ differs from the usual φ-mixing function since we divide by the size of the dependence set of the event E_{A_1}. This is natural in the context of Gibbsian random fields, where the classical φ-mixing mostly fails (except for the simplest i.i.d. case and ad-hoc examples of independent copies of one-dimensional Gibbs measures).

We are now ready to formulate the non-uniformly exponentially φ-mixing (NUEM) condition:

Definition 3.1 (NUEM). *A random field is* **non-uniformly exponentially φ-mixing** *(NUEM) if there exist constants $C, c > 0$ such that*

$$\varphi(m) \leq C \exp(-cm) \quad \textit{for all} \quad m > 0. \tag{3.2}$$

Examples of random field satisfying the NUEM condition are Gibbs measures with exponentially decaying potential in the Dobrushin uniqueness regime, or local transformations of such measures. Of course, for site percolation, where we have independence, we have $\varphi = 0$.

We next define the finite energy property:

Definition 3.2 (Finite energy property). *A probability measure $\mathbb{P}$ has the finite energy property if there exists $\delta \in (0, 1)$ such that*

$$\delta \leq \inf_{\omega \in \Omega} \mathbb{P}(\omega_x = 1 | \omega_{\mathbb{Z}^d \setminus \{x\}}) \leq \sup_{\omega \in \Omega} \mathbb{P}(\omega_x = 1 | \omega_{\mathbb{Z}^d \setminus \{x\}}) \leq 1 - \delta \tag{3.3}$$

Gibbs measures have the finite energy property (in particular, it holds of course for independent site percolation, for which (3.3) holds with $\delta = 1 - \delta = p$), but in general it suffices that there exists a bounded version of $\log \mathbb{P}(\sigma_0 = 1 | \sigma_{\{0\}^c})$. A direct consequence of (3.3) is the existence of $C, C' > 0$ such that for any $\sigma \in \Omega$, $V \subseteq \mathbb{Z}^d$,

$$e^{-C|V|} \leq \mathbb{P}(\omega_V = \sigma_V) \leq e^{-C'|V|}. \tag{3.4}$$

We finally define what it means for a measure to have subcritical clusters:

Definition 3.3 (Sub- and supercritical clusters). (i) *The probability measure $\mathbb{P}$ is said to have subcritical clusters if $\mathbb{P}(|\mathcal{C}_{\mathrm{le}}(0)| < \infty) = 1$ and if there exists $\zeta, \xi \in (0, \infty)$ such that*

$$e^{-\zeta} \leq \liminf_{n \to \infty} \frac{\mathbb{P}(|\mathcal{C}_{\mathrm{le}}(0)| \geq n+1)}{\mathbb{P}(|\mathcal{C}_{\mathrm{le}}(0)| \geq n)} \leq \limsup_{n \to \infty} s \frac{\mathbb{P}(|\mathcal{C}_{\mathrm{le}}(0)| \geq n+1)}{\mathbb{P}(|\mathcal{C}_{\mathrm{le}}(0)| \geq n)} \leq e^{-\xi}. \tag{3.5}$$

(ii) *The probability measure $\mathbb{P}$ is said to have supercritical clusters if $\mathbb{P}(|\mathcal{C}_{\text{le}}(0)| = \infty) > 0$ and if*

$$\lim_{n\to\infty} \frac{\mathbb{P}(n+1 \le |\mathcal{C}_{\text{le}}(0)| < \infty)}{\mathbb{P}(n \le |\mathcal{C}_{\text{le}}(0)| < \infty)} = 1. \tag{3.6}$$

3.2 The exponential law

In order to have the exponential law, we need that the events E_n are somewhat localized. More precisely, the non-occurrence of the event in a big cube can be decomposed as an intersection of non-occurrence of the event in a union of small sub-cubes separated by corridors. Then mixing can be used to factorize the probabilities of on-occurrence in the sub-cubes, provided the corridors are sufficiently large. Optimization of this philosophy is the content of the Iteration Lemma in [2]. In our case, the events are not strictly localized but they can be replaced by local events, without affecting limit laws. This is made precise in the following definition:

Definition 3.4 (Localizability). (i) *Let E_n be a sequence of events such that $\mathbb{P}(E_n) \to 0$. The events are called local w.r.t. $\mathbb{P}(E_n)$, if $E_n \in \mathcal{F}_{k_n}$ with $k_(2n+1)^d \mathbb{P}(E_n)^\theta \to 0$ for any $\theta > 0$.*

(ii) *The events E_n of point 1 are called localizable if there exist events E'_n which are local w.r.t. $\mathbb{P}(E_n)$ such that $\mathbb{P}(E_n) - \mathbb{P}(E'_n) \to 0$ and for any sequence $u_n \to \infty$*

$$\lim_{n\to\infty} |\mathbb{P}(\mathbf{t}_{E'_n} \le u_n) - \mathbb{P}(\mathbf{t}_{E_n} \le u_n)| = 0$$

E'_n is then called a local version of E_n.

We will use the following theorem which can be derived from [2], as we explain below.

Theorem 3.5 (Exponential law). *Suppose is $\mathbb{P}$ is finite energy and satisfies the NUEM condition. Suppose further that E_n are localizable measurable events such that for some $\delta, \gamma > 0$ and all $n \in \mathbb{N}$, $\mathbb{P}(E_n) \le e^{-\gamma n^\delta}$. Assume furthermore that for any $\alpha > 1$*

$$\limsup_{n\to\infty} \sum_{0<|x|\le n^\alpha} \frac{\mathbb{P}(E_n \cap \theta_x E_n)}{\mathbb{P}(E_n)} < \infty \tag{3.7}$$

then there exists $\Lambda_1, \Lambda_2, c, \rho \in (0,\infty)$, such that for all $n \in \mathbb{N}$ there exists $\lambda_{E_n} \in [\Lambda_1, \Lambda_2]$ such that

$$\left| \mathbb{P}\left(\mathbf{t}_{E_n} > \frac{t}{\lambda_{E_n} \mathbb{P}(E_n)} \right) - e^{-t} \right| \le \mathbb{P}(E_n)^\rho e^{-ct} \tag{3.8}$$

For the "local version" E'_n, the theorem follows from [2, Theorem 2.6 and Remark 2.8]. The extension to E_n is straightforward from Definition 3.4 and is formulated in detail in [2, Remark 4.13]. Note that there is some notational difference between the present paper and [2], since in [2], the occurrence time $\mathbf{t}_{E_n}$ is the *width* of the first cube where E_n occurs, whereas in our setting, it is the *volume*.

Condition (3.7) is needed to apply Lemma 4.6 in [2], see also [1]. It ensures the existence of the lower bound Λ_1 on the parameter λ_{E_n} (which is obtained via a second moment estimate for the number of occurrences). It guarantees further that the parameter is bounded away from zero which means that in a cube of volume $\mathbb{P}(E_n)^{-1}$, the event E_n happens with a probability bounded away from zero (uniformly in n). This means that $\mathbb{P}(E_n)^{-1}$ is the right scale, i.e., a cube with this volume is such that the event E_n happens with probability bounded away from zero or one.

The parameter λ_{E_n} measures the "self-repetitive" nature of the event E_n, i.e., whether the event appears typically isolated or in clusters. See also [1] for one-dimensional examples of $\lambda_{E_n} \neq 1$ and conditions ensuring $\lambda_{E_n} = 1$. For the events E_n of our paper, we will show that $\lambda_{E_n} = 1$.

3.3 Main results

In our context, Condition (3.7) is satisfied as soon as for any $\alpha > 1$, we have

$$\limsup_{n\to\infty} \sum_{0<|x|<n^\alpha} \frac{\mathbb{P}\left(\{n \le |\mathcal{C}_{\mathrm{le}}(x)| < \infty\} \cap \{n \le |\mathcal{C}_{\mathrm{le}}(0)| < \infty\}\right)}{\mathbb{P}(n \le |\mathcal{C}_{\mathrm{le}}(0)| < \infty)} < \infty. \tag{3.9}$$

The value of α which we will need later is related to the localization of the event $|\mathcal{C}_{\mathrm{le}}(0)| > n$ to the event $n < |\mathcal{C}_{\mathrm{le}}(0)| < n^\alpha$ (see the proof in Section 4 for more details).

Now we can state our main result for the subcritical case:

Theorem 3.6 (Subcritical Gumbel law). *Suppose $\mathbb{P}$ is finite energy, NUEM, has subcritical clusters and satisfies (3.9). Then there exists a sequence $u_n \to \infty$, and a bounded sequence $a_n \in [e^{-\zeta}, 1]$, such that for $x \ge 0$*

$$e^{-a_n e^{-x\zeta}} \le \mathbb{P}(\mathcal{M}_n \le u_n + x) \le e^{-a_n e^{-x\xi}}. \tag{3.10}$$

When $x < 0$, the upper and lower bound are reversed.

Moreover, if $\xi = \zeta$, then there exists a constant $\rho > 0$ such that

$$|\mathbb{P}\left(\mathcal{M}_n \le u_n + x\right) - e^{-a_n e^{-\zeta x}}| \le \frac{1}{n^\rho}. \tag{3.11}$$

We now turn to examples where we can apply Theorem 3.6. The following proposition yields a class of non-trivial examples:

Proposition 3.7. *If $\mathbb{P}$ is a subcritical Markov measure satisfying the FKG inequality, then (3.9) is satisfied.*

This gives the following applications:

1. Subcritical site percolation $\mathbb{P} = \mathbb{P}_p$ where $\mathbb{P}_p$ is the Bernoulli measure with $\mathbb{P}_p(\omega_0) = p$ and $p < p_c$.
2. In $d = 2$: Ising model at $\beta < \beta_c$. In general dimension, Ising model at high temperature and/or high enough magnetic field (see [22]).

In very general context we have (3.9) in high enough magnetic field. The idea is that as soon as for any V, and any $\omega \in \Omega$, the conditional probabilities $\mathbb{P}_V(\cdot|\omega_{V^c})$ can be dominated by a Bernoulli measure with subcritical clusters, then of course, for any $x \neq 0$,

$$\mathbb{P}\big(|\mathcal{C}_{\mathrm{le}}(x)| \ge n \big| |\mathcal{C}_{\mathrm{le}}(0)| \ge n\big) \le \mathbb{P}_p(|\mathcal{C}_{\mathrm{le}}(0)| \ge n), \tag{3.12}$$

and hence (3.9) is satisfied.

We will now formulate another class of examples. We say that $\mathbb{P}$ is dominated by a Bernoulli measure in the sense of Holley, if for all $\omega \in \Omega$

$$\mathbb{P}(\omega_0 = 1|\omega_{\{0\}^c}) < p. \tag{3.13}$$

This condition implies that $\mathbb{P}$ is stochastically dominated by the Bernoulli measure $\mathbb{P}_p$. For measures that are dominated by a subcritical Bernoulli measure, our results also apply:

Proposition 3.8. *Let p_c denote the critical value for Bernoulli site percolation. If (3.13) is satisfied for some $p < p_c$, then (3.9) holds true.*

This proposition can be applied to Gibbs measures such that the potential has a Dobrushin norm which is small enough (to guarantee mixing condition), with magnetic field high enough such that (3.13) holds, see [22] for more details.

Our last theorem applies for independent supercritical site percolation. Recall (1.15). Then we have the following result for supercritical site percolation:

Theorem 3.9 (Supercritical Gumbel law). *Let $p > p_c$. There exists a constant $a > 0$, a sequence $a_n \in (a, 1]$ and a sequence $u_n(x)$ such that $u_n(x) \uparrow \infty$ for all x as $n \uparrow \infty$, such that for all x*

$$\mathbb{P}_p(\mathcal{M}_n \leq u_n(x)) = e^{-a_n e^{-x}} + o(1). \tag{3.14}$$

If $\mathbb{P}$ has finite supercritical clusters, then $a_n = 1 + o(1)$.

4 Proofs

In this section, we prove the main results stated in Sections 1 and 3.

4.1 Preparations

In this section, we state some general results for non-critical clusters. In Lemma 4.1, we first identify the law of $|\mathcal{C}_{\rm le}(0)|$ for non-critical clusters in terms of the law of $|\mathcal{C}(0)|$. In Proposition 4.2 and Lemma 4.3, we investigate the cluster size distribution in more detail.

Lemma 4.1 (The law of $|\mathcal{C}_{\rm le}(0)|$). *Suppose that $\mathbb{E}_p\left(|\mathcal{C}(0)| I[|\mathcal{C}(0)| < \infty]\right) < \infty$. Then, for $n \geq 1$,*

$$\mathbb{P}(|\mathcal{C}(0)| = n) = n\mathbb{P}(|\mathcal{C}_{\rm le}(0)| = n). \tag{4.1}$$

Proof. We start with the subcritical case. Let V_k be a sequence of volumes such that $V_k \to \mathbb{Z}^d$ and $|\partial V_k|/|V_k| \to 0$ as $k \to \infty$. For a cluster $\mathcal{C}_{\rm le}$, we denote by $le(\mathcal{C}_{\rm le})$ the left endpoint of $\mathcal{C}_{\rm le}$, in particular $le(\mathcal{C}_{\rm le}(x)) = x$ by definition. Then, for $n \geq 1$,

$$\begin{aligned}
|V_k| n\mathbb{P}(|\mathcal{C}_{\rm le}(0)| = n) &= \sum_{x \in V_k} n\mathbb{P}(|\mathcal{C}_{\rm le}(x)| = n) \\
&= \sum_{x \in V_k} \sum_{y} \mathbb{P}(|\mathcal{C}_{\rm le}(x)| = n, y \in \mathcal{C}_{\rm le}(x)) \\
&= \sum_{x \in V_k} \sum_{y} \mathbb{P}\left(|\mathcal{C}(y)| = n, x = le(\mathcal{C}(y))\right) \\
&= \sum_{y} \mathbb{P}\left(|\mathcal{C}(y)| = n, le(\mathcal{C}(y)) \in V_k\right) \\
&= |V_k|\mathbb{P}(|\mathcal{C}(0)| = n) + \sum_{y \notin V_k} \mathbb{P}\left(|\mathcal{C}(y)| = n, le(\mathcal{C}(y)) \in V_k\right) \\
&\qquad - \sum_{y \in V_k} \mathbb{P}\left(|\mathcal{C}(y)| = n, le(\mathcal{C}(y)) \notin V_k\right).
\end{aligned} \tag{4.2}$$

We claim that the last two terms are $O(|\partial V_k|)$. Indeed, the first sum is equal to

$$\sum_{y\notin V_k}\sum_{x\in V_k}\mathbb{P}(|\mathcal{C}_{\text{le}}(x)|=n, y\in\mathcal{C}_{\text{le}}(x))\le\mathbb{E}\Big|\bigcup_{x\in V_k:\mathcal{C}_{\text{le}}(x)\cap V_k^c\neq\varnothing}\mathcal{C}_{\text{le}}(x)\Big|\le\sum_{z\in\partial V_k}\mathbb{E}|\mathcal{C}(z)|=O(|\partial V_k|),\quad(4.3)$$

where the last step follows because $\mathbb{P}$ has subcritical clusters, and hence $\mathbb{P}(|\mathcal{C}(x)|\ge n)\le e^{-\zeta n}$ by [23, Theorem (6.78)], so that $\mathbb{E}|\mathcal{C}(z)|<\infty$ for all z. The second sum is bounded similarly as

$$\sum_{y\in V_k}\sum_{x\notin V_k}\mathbb{P}(|\mathcal{C}_{\text{le}}(x)|=n, y\in\mathcal{C}_{\text{le}}(x))\le\mathbb{E}\Big|\bigcup_{x\notin V_k:\mathcal{C}_{\text{le}}(x)\cap V_k\neq\varnothing}\mathcal{C}_{\text{le}}(x)\Big|\le\sum_{z\in\partial V_n^c}\mathbb{E}|\mathcal{C}(z)|=O(|\partial V_k|).\quad(4.4)$$

Therefore, we obtain that

$$|V_k|n\mathbb{P}(|\mathcal{C}_{\text{le}}(0)|=n)=|V_k|\mathbb{P}(|\mathcal{C}(0)|=n)+O(|\partial V_k|).\qquad(4.5)$$

Divide by $|\partial V_k|$ and let $k\to\infty$ to arrive at the claim.

The proof for the supercritical case is similar and based on the estimate $\mathbb{P}(n\le|\mathcal{C}(x)|<\infty)\le e^{-\gamma n^\delta}$ from which we conclude $\mathbb{E}(|\mathcal{C}(z)|I[|\mathcal{C}(z)|<\infty])<\infty$. □

Before we formulate our next proposition, we remark that the cluster $\mathcal{C}_{\text{le}}(0)$ is finite with probability one, since it has 0 as its left endpoint. Therefore, we have $\mathbb{P}(n\le|\mathcal{C}_{\text{le}}(0)|<\infty)=\mathbb{P}(|\mathcal{C}_{\text{le}}(0)|\ge n)$ in the supercritical case, and we can drop the restriction that the cluster is finite in the notation. Naturally, we also drop this restriction in the subcritical case.

Proposition 4.2 (Lower bound on the cluster tail). *If $\mathbb{P}$ is finite energy, then there exists a $\zeta>0$ such that*

$$\liminf_{n\to\infty}\frac{\mathbb{P}(|\mathcal{C}_{\text{le}}(0)|\ge n+1)}{\mathbb{P}(|\mathcal{C}_{\text{le}}(0)|\ge n)}\ge e^{-\zeta}.\qquad(4.6)$$

Proof. We start with the subcritical case. We abbreviate $X=|\mathcal{C}_{\text{le}}(0)|$. Define $A_n=\{X\ge n\}$. The ratio we are interested in can be written as

$$\frac{\mathbb{P}(X\ge n+1)}{\mathbb{P}(X\ge n)}=\frac{\int_{A_{n+1}}d\mathbb{P}}{\int_{A_{n+1}}d\mathbb{P}+\int_{A_n\setminus A_{n+1}}d\mathbb{P}}\qquad(4.7)$$

For $\omega\in\Omega$, we denote by ω^x the configuration obtained by flipping at x, i.e., $\omega^x(y)=\omega(y)$ for $y\neq x$ and $\omega^x(x)=1-\omega(x)$. Then, for any $\omega\in A_n\setminus A_{n+1}$ there exists $x=x_\omega\in\mathbb{Z}^d$ such that $\omega^x\in A_{n+1}$, which gives

$$\frac{\int_{A_{n+1}}d\mathbb{P}}{\int_{A_{n+1}}d\mathbb{P}+\int_{A_n\setminus A_{n+1}}d\mathbb{P}}\ge\frac{\int_{A_{n+1}}d\mathbb{P}}{\int_{A_{n+1}}d\mathbb{P}+\int_{A_{n+1}}d\mathbb{P}^{x_\omega}}\qquad(4.8)$$

where for $x\in\mathbb{Z}^d$, $\mathbb{P}^x$ denotes the image measure under the transformation $\Psi_x:\omega\mapsto\omega^x$. The finite energy property implies that

$$C=\sup_x\|\frac{d\mathbb{P}^x}{d\mathbb{P}}\|_{L^\infty(\mathbb{P})}<\infty.\qquad(4.9)$$

Therefore, from (4.8), we obtain the lower bound

$$\frac{\mathbb{P}(X\ge n+1)}{\mathbb{P}(X\ge n)}\ge\frac{1}{1+C},$$

which is (4.6).

To deal with the supercritical case, let $\omega \in A_n \setminus A_{n+1}$. Flipping one occupation variable at the exterior boundary of $\mathcal{C}(0)$ can lead to an infinite cluster, so that $\omega^x \notin A_{n+1}$. However, we can make one site occupied and all its neighbors which do not belong to $\mathcal{C}_{\mathrm{le}}(0)$ vacant. This leads to a configuration in $A_{n+1} = \{n+1 \leq |\mathcal{C}_{\mathrm{le}}(0)| < \infty\}$. Since this transformation T_x of ω is still local (it affects only one site $x = x_\omega$ and possibly some neighbors), the same argument applies where now, using the finite energy property we replace C of (4.9) by

$$C' = \sup_x \| \frac{d\mathbb{P} \circ T_x}{d\mathbb{P}} \|_{L^\infty(\mathbb{P})} < \infty. \tag{4.10}$$

□

Lemma 4.3 (Existence of v_n). *There exists $v_n \uparrow \infty$ and a sequence b_n satisfying $0 < b_n \leq 1$ such that*

$$\mathbb{P}(|\mathcal{C}_{\mathrm{le}}(0)| > v_n) = \frac{b_n}{n}. \tag{4.11}$$

It $\mathbb{P}$ has subcritical clusters, then $\liminf_{n\to\infty} b_n \geq e^{-\zeta}$, while if $\mathbb{P}$ has finite supercritical clusters, then $b_n = 1 + o(1)$.

Proof. We abbreviate $X = |\mathcal{C}_{\mathrm{le}}(0)|$, and recall that $X < \infty$ both above and below criticality. We define

$$\begin{aligned} v_n^+ &= \inf\{x \in \mathbb{N} : \mathbb{P}(X \geq x) \leq \frac{1}{n}\}, \\ v_n^- &= \sup\{x \in \mathbb{N} : \mathbb{P}(X \geq x) \geq \frac{1}{n}\}. \end{aligned} \tag{4.12}$$

Then $v_n^+ = v_n^- + 1$ or $v_n^+ = v_n^-$. Put $v_n = v_n^+$. By definition

$$\mathbb{P}(X \geq v_n) \leq \frac{1}{n},$$

so that $b_n \leq 1$.

Moreover,

$$\begin{aligned} n\mathbb{P}(X \geq v_n) &= n\mathbb{P}(X \geq v_n^-)\frac{\mathbb{P}(X \geq v_n^+)}{\mathbb{P}(X \geq v_n^-)} \\ &\geq \frac{\mathbb{P}(X \geq v_n^- + 1)}{\mathbb{P}(X \geq v_n^-)}. \end{aligned} \tag{4.13}$$

We note that $v_n \to \infty$ when $n \to \infty$. Therefore, if $\mathbb{P}$ has subcritical or supercritical clusters,

$$\liminf_{n\to\infty} n\mathbb{P}(X \geq v_n) \geq \liminf_{n\to\infty} \frac{\mathbb{P}(X \geq n+1)}{\mathbb{P}(X \geq n)} = e^{-\zeta}. \tag{4.14}$$

Thus, we obtain $\liminf_{n\to\infty} b_n \geq e^{-\zeta}$. On the other hand, when $\mathbb{P}$ has finite supercritical clusters,

$$1 \geq \lim_{n\to\infty} n\mathbb{P}(X \geq v_n) \geq \lim_{n\to\infty} \frac{\mathbb{P}(X \geq n+1)}{\mathbb{P}(X \geq n)} = 1. \tag{4.15}$$

Thus, we obtain $\lim_{n\to\infty} b_n = 1$. □

We next verify that the events

$$E_n = \{|\mathcal{C}_{\rm le}(0)| \geq n\} \tag{4.16}$$

are localizable. This is the content of the next lemma.

Lemma 4.4 (Localizability of E_n). *The events $E_n = \{|\mathcal{C}_{\rm le}(0)| \geq n\}$ are localizable, and their local versions can be chosen as*

$$E'_n = \{n \leq |\mathcal{C}_{\rm le}(0)| < n^\theta\} \tag{4.17}$$

for some $\theta \in (1,\infty)$ with $k_n = n^\theta$ in Definition 3.4.

Proof. This is an obvious consequence of the estimates that there exist positive $\zeta = \zeta(p,d)$ and $\eta = \eta(p,d)$ such that $\mathbb{P}(|\mathcal{C}(0)| > n) \leq e^{-\zeta n}$ in the subcritical case and $\mathbb{P}(n \leq |\mathcal{C}(0)| < \infty) \leq e^{-\eta n^{\frac{d-1}{d}}}$ in the supercritical case, for some $\zeta, \eta > 0$. See [23, Theorems (6.78) and (8.61)] for these estimates in the context of percolation. $\square$

We finish this section with a lemma showing the asymptotic equivalence of τ_{E_n} introduced in (2.5) and the occurrence time $\mathbf{t}_{E_n}$.

More precisely, we have the following lemma:

Lemma 4.5 (Occurence times). *Let $m = m_n \uparrow \infty$ be such that $m_n n^{\epsilon-1}$ converges to zero as $n \to \infty$ for some $\epsilon \in (0,1)$, and such that $\mathbb{P}(E_{m_n}) \leq n^{-d+\epsilon}$. Then*

$$\mathbb{P}(\tau_{E_{m_n}} \leq (2n+1)^d) = \mathbb{P}(\mathbf{t}_{E_{m_n}} \leq (2n+1)^d) + o(1). \tag{4.18}$$

Proof. First we remark that

$$\{\mathbf{t}_{E_m} \leq (2n+1)^d\} \subseteq \{\tau_{E_m} \leq (2n+1)^d\}, \tag{4.19}$$

and

$$\{\tau_{E_m} \leq (2n+1)^d\} \setminus \{\mathbf{t}_{E_m} \leq (2n+1)^d\} \subseteq \{\exists x \in B_n : x + B_m \not\subseteq B_n : |\mathcal{C}_{\rm le}(x)| > m\}. \tag{4.20}$$

We estimate

$$\begin{aligned} \mathbb{P}\left(\{\exists x \in B_n : x + B_{m_n} \not\subseteq B_n : |\mathcal{C}_{\rm le}(x)| > m_n\}\right) &\leq \mathbb{P}(|\mathcal{C}_{\rm le}(0)| \geq m_n)|\{x \in B_n : x + B_{m_n} \not\subseteq B_n\}| \\ &\leq \frac{m_n n^{d-1}}{n^{d-\epsilon}} = m_n n^{\epsilon-1}. \end{aligned} \tag{4.21}$$

This converges to zero as $n \to \infty$ by the assumption on m_n. $\square$

4.2 Maximal subcritical clusters

In this section, we prove Theorems 1.1 and 3.6. We study the tails of the cluster size distribution, subject to (3.5). The main result is the following lemma:

Lemma 4.6 (Identification $u_n(x)$). *Suppose $\mathbb{P}$ has finite energy and has subcritical clusters, then there exists a sequence a_n with $0 < a_n \leq 1$ such that $\liminf_{n\to\infty} a_n \geq e^{-\zeta}$, such that for all $x > 0$ and for all $n \geq 1$,*

$$\frac{a_n}{(2n+1)^d} e^{-\zeta x}(1+o(1)) \leq \mathbb{P}(|\mathcal{C}_{\rm le}(0)| \geq u_n + x) \leq \frac{a_n}{(2n+1)^d} e^{-\xi x}(1+o(1)). \tag{4.22}$$

For $x < 0$ the same inequality holds with ζ and ξ interchanged.

Proof. Let $x > 0$. We again abbreviate $X = |\mathcal{C}_{\mathrm{le}}(0)|$.

$$\mathbb{P}(X \geq u_n + x) = \prod_{i=1}^{x} \frac{\mathbb{P}(X \geq u_n + i)}{\mathbb{P}(X \geq u_n + i - 1)}. \tag{4.23}$$

Hence, with the choice of $u_n = v_{(2n+1)^d}$ where v_n is as in Lemma 4.3,

$$\liminf_{n\to\infty} n\mathbb{P}(X \geq u_n + x) \geq \liminf_{n\to\infty} n\mathbb{P}(X \geq u_n) e^{-\zeta x}, \tag{4.24}$$

and, using Lemma 4.3 again, for any $x \in \mathbb{N}$ fixed,

$$\limsup_{n\to\infty} n\mathbb{P}(X \geq u_n + x) \leq \left(\limsup_{n\to\infty} \frac{\mathbb{P}(X \geq n+1)}{\mathbb{P}(X \geq n)} \right)^x = e^{-\xi x}. \tag{4.25}$$

This proves the claim for $x > 0$. The proof for $x < 0$ is similar. $\square$

We now verify Condition (3.7) for FKG measures.

Proof of Proposition 3.7. We have to prove that for any $\alpha > 0$,

$$\limsup_{n\to\infty} \sum_{0<|x|\leq n^\alpha} \frac{\mathbb{P}(|\mathcal{C}_{\mathrm{le}}(0)| \geq n, |\mathcal{C}_{\mathrm{le}}(x)| \geq n)}{\mathbb{P}(|\mathcal{C}_{\mathrm{le}}(0)| \geq n)} < \infty. \tag{4.26}$$

In fact, we will show that the right-hand side of (4.26) converges to 0 when $n \to \infty$.

We denote by $\mathbb{P}^\eta_\Lambda(\omega_\Lambda)$ the conditional probability to find ω inside Λ, given η outside Λ. For a Markov random field, the dependence on η is only through the boundary of Λ, i.e.,

$$\mathbb{P}^\eta_\Lambda(\omega_\Lambda) = \mathbb{P}^{\eta_{\partial\Lambda}}_\Lambda(\omega_\Lambda), \tag{4.27}$$

where $\partial\Lambda$ denotes the exterior boundary of Λ, i.e., the set of those sites not belonging to Λ which have at least one neighbor inside Λ. Thus, we can think of η as describing the boundary condition. By the FKG-property, we have that if $\eta \leq \zeta$ for $\eta, \zeta \in \{0,1\}^{\mathbb{Z}^d}$, then

$$\mathbb{P}^\eta_\Lambda \leq \mathbb{P}^\zeta_\Lambda \tag{4.28}$$

Moreover, by definition of the clusters $\mathcal{C}_{\mathrm{le}}(x)$, $\mathcal{C}_{\mathrm{le}}(0) \cap \mathcal{C}_{\mathrm{le}}(x) = \varnothing$ for $x \neq 0$. Therefore, we can write, for $x \neq 0$,

$$\begin{aligned}
&\mathbb{P}(|\mathcal{C}_{\mathrm{le}}(0)| \geq n, |\mathcal{C}_{\mathrm{le}}(x)| \geq n) \\
&\quad = \sum_{A:|A|\geq n, le(A)=0} \mathbb{P}(\mathcal{C}_{\mathrm{le}}(0) = A, |\mathcal{C}_{\mathrm{le}}(x)| \geq n, \mathcal{C}_{\mathrm{le}}(x) \cap A = \varnothing) \\
&\quad = \sum_{A:|A|\geq n, le(A)=0} \mathbb{P}(|\mathcal{C}_{\mathrm{le}}(x)| \geq n, \mathcal{C}_{\mathrm{le}}(x) \cap A = \varnothing | \omega_A = 1, \omega_{\partial A} = 0) \mathbb{P}(\mathcal{C}_{\mathrm{le}}(0) = A) \\
&\quad \leq \sum_{A:|A|\geq n, le(A)=0} \mathbb{P}(\mathcal{C}_{\mathrm{le}}(0) = A) \mathbb{P}^{0_{\partial A}}_{\mathbb{Z}^d \setminus \bar{A}}(|\mathcal{C}_{\mathrm{le}}(x)| \geq n),
\end{aligned} \tag{4.29}$$

where in the last step we have used the Markov property, with the notation $\bar{A} = A \cup \partial A$. Using (4.28), we thus arrive at

$$\mathbb{P}^{0_{\partial A}}_{\mathbb{Z}^d \setminus \bar{A}}(n \leq |\mathcal{C}_{\mathrm{le}}(x)| \leq n^\theta) \leq \mathbb{P}_{\mathbb{Z}^d \setminus \bar{A}}(|\mathcal{C}(x)| \geq n) \leq \mathbb{P}(|\mathcal{C}(x)| \geq n). \tag{4.30}$$

Equation (4.30) combined with (4.29) leads to the correlation inequality

$$\mathbb{P}(|\mathcal{C}_{\mathrm{le}}(0)| \geq n, |\mathcal{C}_{\mathrm{le}}(x)| \geq n) \leq \mathbb{P}(|\mathcal{C}_{\mathrm{le}}(0)| \geq n)\mathbb{P}(|\mathcal{C}(0)| \geq n). \tag{4.31}$$

Therefore,

$$\sum_{0<|x|\leq n^\alpha} \frac{\mathbb{P}(|\mathcal{C}_{\mathrm{le}}(0)| \geq n, |\mathcal{C}_{\mathrm{le}}(x)| \geq n)}{\mathbb{P}(|\mathcal{C}_{\mathrm{le}}(0)| \geq n)} \leq (2n^\alpha + 1)^d \mathbb{P}(|\mathcal{C}(0)| \geq n) \to 0, \tag{4.32}$$

because the decay of the probability $\mathbb{P}(|\mathcal{C}(0)| \geq n)$ is faster than $\frac{1}{n^\beta}$ for any $\beta > 0$. □

Proposition 4.7 (The subcritical intensity is one). *For u_n as in Lemma 4.3 and for every x bounded, there exists a $\beta > 0$ such that*

$$1 - \mathbb{P}(E_{u_n+x})^\beta \leq \lambda_{E_{u_n+x}} \leq 1. \tag{4.33}$$

Proof. We will first identify $\lambda_{E_{u_n+x}}$. We use [2, (2.6)], which states that

$$\lambda_E = -\frac{\log \mathbb{P}\big(\mathbf{t}_E > f_E\big)}{f_E \mathbb{P}(E)}, \tag{4.34}$$

where, for some $\gamma \in (0, 1)$

$$f_E = \lfloor \mathbb{P}(E)^{-\gamma} \rfloor. \tag{4.35}$$

We will show that $\mathbb{P}\big(\mathbf{t}_E \leq f_E\big)$ is quite small (as proved in the sequel), so that we can approximate

$$-\log \mathbb{P}\big(\mathbf{t}_E > f_E\big) = \mathbb{P}\big(\mathbf{t}_E \leq f_E\big) + O\Big(\mathbb{P}\big(\mathbf{t}_E \leq f_E\big)^2\Big). \tag{4.36}$$

Therefore,

$$\lambda_E = \frac{\mathbb{P}\big(\mathbf{t}_E \leq f_E\big)}{f_E \mathbb{P}(E)} + o(1). \tag{4.37}$$

We will proceed by computing $\mathbb{P}\big(\mathbf{t}_E \leq f_E\big)$. To do so, we write

$$\mathbb{P}\big(\mathbf{t}_{E_{u_n+x}} \leq f_{E_{u_n+x}}\big) = \mathbb{P}\Big(\bigcup_{y \in B_{m_{n,x}}} \{|\mathcal{C}_{\mathrm{le}}(y)| \geq u_n + x\}\Big), \tag{4.38}$$

where we abbreviate $m_{n,x} = f_{E_{u_n+x}}^{1/d}$. By Boole's inequality,

$$\mathbb{P}\big(\mathbf{t}_{E_{u_n+x}} \leq f_{E_{u_n+x}}\big) \leq \sum_{y \in B_{m_{n,x}}} \mathbb{P}(|\mathcal{C}_{\mathrm{le}}(y)| \geq u_n + x) = f_{E_{u_n+x}} \mathbb{P}(E_{u_n+x}). \tag{4.39}$$

Thus,

$$\lambda_{E_{u_n+x}} \leq 1. \tag{4.40}$$

For the lower bound, use

$$\begin{aligned} \mathbb{P}\big(\mathbf{t}_{E_{u_n+x}} \leq f_{E_{u_n+x}}\big) \ \geq \ & \sum_{y \in B_{m_{n,x}}} \mathbb{P}(|\mathcal{C}_{\mathrm{le}}(y)| \geq u_n + x) \\ & - \sum_{y,z \in B_{m_{n,x}} : y \neq z} \mathbb{P}(|\mathcal{C}_{\mathrm{le}}(y)| \geq u_n + x, |\mathcal{C}_{\mathrm{le}}(z)| \geq u_n + x). \end{aligned} \tag{4.41}$$

The first term is identical to the first term in the upper bound, and we need to bound the second term only. For this, we use (4.31), and thus obtain

$$\mathbb{P}\big(\mathbf{t}_{E_{u_n+x}} \leq f_{E_{u_n+x}}\big) \geq f_{E_{u_n+x}}\mathbb{P}(E_{u_n+x}) - f^2_{E_{u_n+x}}\mathbb{P}(E_{u_n+x})\mathbb{P}(|\mathcal{C}(0)| \geq u_n + x). \tag{4.42}$$

Thus,

$$\lambda_{E_{u_n+x}} \geq 1 - f_{E_{u_n+x}}\mathbb{P}(|\mathcal{C}(0)| \geq u_n + x) \geq 1 - \mathbb{P}(E_{u_n+x})^{\beta} \tag{4.43}$$

for some $\beta > 0$. □

We finally identify the sequence u_n under the hypothesis of a "classical" subcritical cluster tail behavior in Proposition 4.8, and under the hypothesis of a "classical" supercritical cluster tail behavior in Proposition 4.12.

Proposition 4.8 (Identification $u_n(x)$ for classical subcritical tails). *Suppose that there exists $\alpha \in \mathbb{R}, \zeta > 0$ and $0 < C < \infty$, such that*

$$\mathbb{P}(|\mathcal{C}_{\mathrm{le}}(0)| \geq n) = Cn^{\alpha}e^{-\zeta n}[1 + o(1)]. \tag{4.44}$$

Then

$$u_n = \Big\lfloor \frac{\log n}{\zeta} + \frac{\alpha \log\log n}{\zeta} \Big\rfloor. \tag{4.45}$$

Proof. This is a simple computation, using the definition of u_n introduced in the proof of Lemma 4.3. □

Proof of Theorem 3.6 and Theorem 1.1. We first finish the proof of Theorem 3.6. We first use the equality

$$\{\mathcal{M}_n \geq m\} = \{\tau_{E_m} \leq (2n+1)^d\}. \tag{4.46}$$

Then we use Lemma 4.5 to obtain that as long as $\mathbb{P}(E_{u_n(x)}) \leq n^{-d+\epsilon}$, we have

$$\mathbb{P}(\mathcal{M}_n \geq u_n(x)) = \mathbb{P}(\mathbf{t}_{E_{u_n(x)}} \leq (2n+1)^d) + o(1). \tag{4.47}$$

We wish to apply Theorem 3.8, and will first check that the conditions are fulfilled. We note from Lemma 4.4 that the events $E_{u_n(x)}$ are localizable with local versions $E'_{u_n(x)}$. Furthermore, from Proposition 3.7, it follows that Condition (3.7) is fulfilled for $E_{u_n(x)}$. Therefore, we may apply Theorem 3.8.

We choose $u_n(x) = u_n + x$ as in Lemma 4.9, and the event E_{u_n+x} as before. Note that for this $u_n(x)$, we indeed have that for every x fixed,

$$\mathbb{P}(E_{u_n+x}) = \frac{e^{-x}}{(2n+1)^d}a_n \leq n^{-d+\epsilon}, \tag{4.48}$$

so that we can use (4.47).

Assume that $x \geq 0$. For $x < 0$ some inequalities reverse sign. Then we apply Theorem 3.8 to obtain:

$$\mathbb{P}(\mathcal{M}_n \geq u_n + x) = \mathbb{P}(\mathbf{t}_{E_{u_n+x}} \leq (2n+1)^d) + o(1) = 1 - \exp\big(-\lambda_{E_{u_n+x}}(2n+1)^d\mathbb{P}(E_{u_n+x})\big) + o(1). \tag{4.49}$$

We need to investigate the exponent. By Lemma 4.6, we have that

$$\frac{a_n}{(2n+1)^d} e^{-\zeta x} \leq \mathbb{P}(E_{u_n+x}) \leq \frac{a_n}{(2n+1)^d} e^{-\xi x}, \tag{4.50}$$

and this inequality is reversed for $x < 0$. By Proposition 4.7, we have that

$$\lambda_{E_{u_n+x}} = 1 + o(1). \tag{4.51}$$

Therefore, for any $x \in \mathbb{N}$,

$$1 - \exp(-a_n e^{-\xi x}) + o(1) \leq \mathbb{P}(\mathcal{M}_n \geq u_n + x) \leq 1 - \exp(-a_n e^{-\xi x}) + o(1). \tag{4.52}$$

This completes the proof of Theorem 3.6. When $\zeta = \xi$, the statement in Theorem 1.1 is a direct consequence of Theorem 3.6, combined with Lemma 4.5.

□

Remark. The examples mentioned in Section 1.3.2 fit into the context of Theorem 3.6. Indeed, for the Ising model, the inequality (3.5) is verified above the critical temperature in $d = 2$ and at high enough temperature in any dimension. The mixing condition (3.2) is verified at high temperature in the Dobrushin uniqueness regime, and in $d = 2$ above the critical temperature, by complete analyticity. For general Gibbs measures with a potential with a finite Dobrushin norm, one can choose the magnetic field high enough such that the Dobrushin uniqueness condition and hence condition (3.2) is satisfied (see e.g. [21]), and such that (3.5) follows from a domination with Bernoulli measures (see [22]).

4.3 Maximal supercritical clusters

In this subsection we prove Theorems 3.9 and 1.2.

In the following proposition we show that we can still find a sequence $u_n(x)$, but not necessarily of the form $u_n + x$, if we omit the subcriticality condition. This will be useful when we study the supercritical percolation clusters.

Lemma 4.9 (Existence of $u_n(x)$). *Suppose $\mathbb{P}$ is finite energy, NUEM and $\mathbb{E}(|\mathcal{C}(0)|I[|\mathcal{C}(0)| < \infty]) < \infty$. Then there exists a function $u_n(x)$ such that*

$$\mathbb{P}(|\mathcal{C}_{\mathrm{le}}(0)| \geq u_n(x)) = \frac{e^{-x}}{(2n+1)^d} a_n, \tag{4.53}$$

where a_n is a bounded sequence (not depending on x). Furthermore, if $\mathbb{P}$ has finite supercritical clusters, then $a_n = 1 + o(1)$.

Proof. Since $\mathbb{E}(|\mathcal{C}(0)|I[|\mathcal{C}(0)| < \infty]) < \infty$, we can use Lemma 4.1. As in the proof of Lemma 4.3, we define

$$\begin{aligned} v_n^+(x) &= \inf\{k : \mathbb{P}(X \geq k) \geq \frac{e^{-x}}{n}\}, \\ v_n^-(x) &= \sup\{k : \mathbb{P}(X \geq k) \leq \frac{e^{-x}}{n}\}. \end{aligned} \tag{4.54}$$

Then we can repeat the proof of Lemma 4.3, and use (3.6) to conclude that $a_n = 1 + o(1)$. We can then choose $u_n = v_{(2n+1)^d}$. □

We continue with the following proposition which will guarantee Condition (3.7) for finite supercritical clusters.

Proposition 4.10 (Supercritical second moment condition). *For every $\alpha > 1$*

$$\limsup_{n} \sum_{0<|x|<n^{\alpha}} \frac{\mathbb{P}\left(E_n \cap \theta_x E_n\right)}{\mathbb{P}\left(E_n\right)} = 0. \tag{4.55}$$

Proof. We rewrite

$$\mathbb{P}\left(E_n \cap \theta_x E_n\right) = \mathbb{P}\left(E_n \cap \theta_x E_n \cap \{x \not\longrightarrow \partial B_{2n^{\alpha}}\}\right) + \mathbb{P}\left(E_n \cap \theta_x E_n \cap \{x \longrightarrow \partial B_{2n^{\alpha}}\}\right). \tag{4.56}$$

The second term is simple, since the event is contained in the probability that $\{|\mathcal{C}(x)| > n^{\alpha}\}$. We bound its contribution to the left-hand side of (4.55) by

$$(n^{\alpha}+1)^{d}\frac{\mathbb{P}\left(E_{n^{\alpha}}\right)}{\mathbb{P}\left(E_n\right)}, \tag{4.57}$$

which is an error for any $\alpha > 1$.

For $\Gamma \subseteq \mathbb{Z}^d$, we denote $\overline{\Gamma} = \Gamma \cup \partial_e \Gamma$. Then, we compute

$$\begin{aligned}
&\mathbb{P}\left(E_n \cap \theta_x E_n \cap \{x \not\longrightarrow \partial B_{2n^{\alpha}}\}\right) \qquad (4.58)\\
&\quad= \mathbb{P}\left(E_n \cap \{|\mathcal{C}_{\mathrm{le}}(x)| \geq n\} \cap \{x \not\longrightarrow \partial B_{2n^{\alpha}}\}\right)\\
&\quad= \sum_{\Gamma \in \mathcal{G}_n(0)} \mathbb{P}(\mathcal{C}_{\mathrm{le}}(0) = \Gamma)\mathbb{P}\left(|\mathcal{C}_{\mathrm{le}}(x)| \geq n, x \not\longrightarrow \partial B_{2n^{\alpha}} | \mathcal{C}_{\mathrm{le}}(0) = \Gamma\right)\\
&\quad= \sum_{\Gamma \in \mathcal{G}_n(0)} \mathbb{P}(\mathcal{C}_{\mathrm{le}}(0) = \Gamma)\mathbb{P}_{\mathbb{Z}^d \setminus \overline{\Gamma}}\left(|\mathcal{C}_{\mathrm{le}}(x)| \geq n, x \not\longrightarrow \partial B_{2n^{\alpha}} | \mathcal{C}_{\mathrm{le}}(0) = \Gamma\right),
\end{aligned}$$

where

$$\mathcal{G}_n(0) = \{\Gamma : 0 = le(\Gamma), n \leq |\Gamma| < \infty\}, \tag{4.59}$$

and where $\mathbb{P}_{\mathbb{Z}^d \setminus \overline{\Gamma}}$ is the conditional measure given that all sites in $\overline{\Gamma}$ are vacant. We further define

$$\mathcal{CR}_x = \cup_{y \in \partial(x+B_{2n^{\alpha}})}\left(\mathcal{C}(y) \cap (x + B_{2n^{\alpha}})\right). \tag{4.60}$$

Then we can further condition on $\mathcal{CR}_x$:

$$\begin{aligned}
&\sum_{\Gamma \in \mathcal{G}_n(0)} \mathbb{P}(\mathcal{C}_{\mathrm{le}}(0) = \Gamma)\mathbb{P}_{\mathbb{Z}^d \setminus \overline{\Gamma}}\left(|\mathcal{C}_{\mathrm{le}}(x)| \geq n, x \not\longrightarrow \partial B_{2n^{\alpha}} | \mathcal{C}_{\mathrm{le}}(0) = \Gamma\right)\\
&= \sum_{\Gamma \in \mathcal{G}_n(0)} \sum_{CR \ldots} \mathbb{P}(\mathcal{C}_{\mathrm{le}}(0) = \Gamma)\mathbb{P}(\mathcal{CR}_x = CR)\mathbb{P}_{\mathbb{Z}^d \setminus (\overline{\Gamma} \cup \overline{CR})}(|\mathcal{C}_{\mathrm{le}}(x)| \geq n)
\end{aligned}$$

where we abbreviated the conditions the set CR has to satisfy by .... We can then proceed, using the FKG inequality

$$\begin{aligned}
&\sum_{\Gamma \in \mathcal{G}_n(0)} \sum_{CR \ldots} \mathbb{P}(\mathcal{C}_{\mathrm{le}}(0) = \Gamma)\mathbb{P}(\mathcal{CR}_x = CR)\mathbb{P}_{\mathbb{Z}^d \setminus (\overline{\Gamma} \cup \overline{CR})}(|\mathcal{C}_{\mathrm{le}}(x)| \geq n)\\
&\quad\leq \sum_{\Gamma \in \mathcal{G}_n(0)} \sum_{CR \ldots} \mathbb{P}(\mathcal{C}_{\mathrm{le}}(0) = \Gamma)\mathbb{P}(\mathcal{CR}_x = CR)\mathbb{P}_{\mathbb{Z}^d \setminus (\overline{CR})}(|\mathcal{C}(x)| \geq n)\\
&\quad\leq \sum_{\Gamma \in \mathcal{G}_n(0)} \mathbb{P}(\mathcal{C}_{\mathrm{le}}(0) = \Gamma)\mathbb{P}(n \leq |\mathcal{C}(x)| < \infty)\\
&\quad= \mathbb{P}\left(E_n\right)\mathbb{P}(n \leq |\mathcal{C}(0)| < \infty). \qquad (4.61)
\end{aligned}$$

By [23, Theorem (8.65)], there exists $\eta = \eta(p, d) > 0$ such that

$$e^{-\gamma n^{\frac{d-1}{d}}} \leq \mathbb{P}\left(E_n\right) \leq \mathbb{P}\left(n \leq |\mathcal{C}(0)| < \infty\right) \leq e^{-\eta n^{\frac{d-1}{d}}}. \tag{4.62}$$

From (4.61) and (4.62), we conclude that there exists $\delta > 0$ such that

$$\begin{aligned}\sum_{0<|x|<n^\alpha} \frac{\mathbb{P}\left(E_n \cap \theta_x E_n\right)}{\mathbb{P}\left(E_n\right)} &\leq (2n^\alpha+1)^d \mathbb{P}(n \leq |\mathcal{C}(0)| < \infty) + (2n^\alpha+1)^d \frac{\mathbb{P}(E_{n^\alpha})}{\mathbb{P}(E_n)} \\ &\leq (2n^\alpha+1)^d e^{-\delta n^{\frac{d-1}{d}}}, \end{aligned} \tag{4.63}$$

and thus (4.55) follows.

$\square$

Proposition 4.11 (The supercritical intensity is one). *For $u_n(x)$ as in Lemma 4.3 and for every x bounded, there exists a $\beta > 0$ such that*

$$1 - \mathbb{P}(E_{u_n(x)})^\beta \leq \lambda_{E_{u_n(x)}} \leq 1. \tag{4.64}$$

Proof. We follow the proof of Proposition 4.7. We will first identify $\lambda_{E_{u_n(x)}}$. Recall (4.34) and (4.37). The upper bound in (4.39) applies verbatim.

For the lower bound, use

$$\begin{aligned}\mathbb{P}\big(\mathbf{t}_{E_{u_n(x)}} \leq f_{E_{u_n(x)}}\big) \geq & \sum_{y \in B_{m_{n,x}}} \mathbb{P}(u_n(x) \leq |\mathcal{C}_{\mathrm{le}}(y)| < \infty) \\ & - \sum_{y,z \in B_{m_{n,x}}: y \neq z} \mathbb{P}(u_n(x) \leq |\mathcal{C}_{\mathrm{le}}(y)| < \infty, u_n(x) \leq |\mathcal{C}_{\mathrm{le}}(z)| < \infty),\end{aligned} \tag{4.65}$$

where now $m_{n,x} = f^{1/d}_{E_{u_n(x)}}$. The first term is identical to the first term in the upper bound, and we need to bound the second term only. In order to do so, we derive a similar bound as in (4.31), which was instrumental in the proof of Proposition 4.7.

Write

$$\begin{aligned}&\mathbb{P}(u_n(x) \leq |\mathcal{C}_{\mathrm{le}}(y)| < \infty, u_n(x) \leq |\mathcal{C}_{\mathrm{le}}(z)| < \infty) \\ &\quad = \mathbb{P}(|\mathcal{C}_{\mathrm{le}}(y)| \leq u_n(x), |\mathcal{C}_{\mathrm{le}}(z)| \leq u_n(x), y, z \not\longrightarrow \partial B_n) \\ &\qquad + \mathbb{P}\big(\{u_n(x) \leq |\mathcal{C}_{\mathrm{le}}(y)| < \infty\} \cap \{u_n(x) \leq |\mathcal{C}_{\mathrm{le}}(z)| < \infty\} \cap \big(\{y \longrightarrow \partial B_n\} \cup \{z \longrightarrow \partial B_n\}\big)\big).\end{aligned} \tag{4.66}$$

The first is bounded by a similar argument as in (4.61) by

$$\begin{aligned}&\mathbb{P}(|\mathcal{C}_{\mathrm{le}}(y)| \geq u_n(x), |\mathcal{C}_{\mathrm{le}}(z)| \geq u_n(x), y, z \not\longrightarrow \partial B_n) \\ &\quad \leq \mathbb{P}(|\mathcal{C}_{\mathrm{le}}(y)| \geq u_n(x), y \not\longrightarrow \partial B_n) \mathbb{P}(|\mathcal{C}(z)| \geq u_n(x), z \not\longrightarrow \partial B_n) \\ &\quad \leq \mathbb{P}(E_{u_n(x)}) \mathbb{P}(u_n(x) \leq |\mathcal{C}(0)| < \infty).\end{aligned} \tag{4.67}$$

Using that $u_n(x) \leq n/2$, the second event in (4.67) is bounded by

$$\mathbb{P}(E_{n-u_n(x)}) \leq \mathbb{P}(E_{n/2}) \leq e^{-cn^{\frac{d-1}{d}}},$$

which is much smaller than $\mathbb{P}(E_{u_n(x)})\mathbb{P}(u_n(x) \leq |\mathcal{C}(0)| < \infty)$. Therefore, we obtain that for $y \neq z$

$$\begin{aligned}&\mathbb{P}(u_n(x) \leq |\mathcal{C}_{\rm le}(y)| < \infty, u_n(x) \leq |\mathcal{C}_{\rm le}(y)| < \infty)\\&\quad \leq \mathbb{P}(E_{u_n(x)})\mathbb{P}(u_n(x) \leq |\mathcal{C}(0)| < \infty)(1+o(1)).\end{aligned} \tag{4.68}$$

We use (4.68), and thus obtain

$$\begin{aligned}\mathbb{P}\big(\mathbf{t}_{E_{u_n(x)}} \leq f_{E_{u_n(x)}}\big) \;\geq\; & f_{E_{u_n(x)}}\mathbb{P}(E_{u_n(x)})\\&-f^2_{E_{u_n(x)}}\mathbb{P}(E_{u_n(x)})\mathbb{P}(u_n(x) \leq |\mathcal{C}(0)| < \infty)(1+o(1)).\end{aligned} \tag{4.69}$$

Thus,

$$\lambda_{E_{u_n(x)}} \geq 1 - f_{E_{u_n(x)}}\mathbb{P}(u_n(x) \leq |\mathcal{C}(0)| < \infty)(1+o(1)) \geq 1 - \mathbb{P}(E_{u_n(x)})^{\beta} \tag{4.70}$$

for some $\beta > 0$. □

Finally, for supercritical clusters we expect that

$$\mathbb{P}(n \leq |\mathcal{C}_{\rm le}(0)| < \infty) = Cn^{\alpha}e^{-\eta n^{\delta}}[1+o(1)], \tag{4.71}$$

i.e., Weibull tails (possibly with polynomial corrections), and with $\delta = \frac{d-1}{d}$.

So far, (4.71) has not been proved rigorously, but if we assume such a tail behavior, then we can infer the precise form of the sequence $u_n(x)$ in Lemma 4.9.

Proposition 4.12 (Identification $u_n(x)$ for classical supercritical tails). *If (4.71) is satisfied, then the sequence $u_n(x)$ can be chosen of the form*

$$u_n(x) = \Big\lfloor \Big(\frac{\log n}{\eta} + \frac{\alpha \log\log n}{\eta\delta} + x\Big)^{1/\delta}\Big\rfloor. \tag{4.72}$$

Proof. Under the condition (4.71), it is a simple computation to verify that

$$\mathbb{P}(u_n(x) \leq |\mathcal{C}_{\rm le}(0)| < \infty) = \frac{e^{-x}}{(2n+1)^d}(1+o(1)). \tag{4.73}$$

□

Proof of Theorem 3.9 and Theorem 1.2. We first finish the proof of Theorem 3.9. We follow the line of argument in the proof of Theorem 3.6.

We first use (4.46). Then we use Lemma 4.5 to obtain that as long as $\mathbb{P}(E_{u_n(x)}) \leq n^{-d+\epsilon}$, we have (4.47).

We again apply Theorem 3.8, and check the conditions. We note from Lemma 4.4 that the events $E_{u_n(x)}$ are localizable with local versions $E'_{u_n(x)}$. Furthermore, from Proposition 4.10, it follows that Condition (3.7) is fulfilled for $E_{u_n(x)}$. Therefore, we may apply Theorem 3.8.

We choose $u_n(x)$ as in Lemma 4.9, and the event $E_{u_n(x)}$ as before. Note that for this $u_n(x)$, we indeed have that

$$\mathbb{P}(E_{u_n(x)}) = \frac{e^{-x}}{(2n+1)^d}a_n \leq n^{-d+\epsilon}, \tag{4.74}$$

so that we can use (4.47).

Assume that $x \geq 0$. For $x < 0$ some inequalities reverse sign. Then we apply Theorem 3.8 to obtain:

$$\mathbb{P}(\mathcal{M}_n \geq u_n(x)) = \mathbb{P}(\mathbf{t}_{E_{u_n(x)}} \leq (2n+1)^d) + o(1) = 1 - \exp\left(-\lambda_{E_{u_n(x)}}(2n+1)^d \mathbb{P}(E_{u_n(x)})\right) + o(1). \tag{4.75}$$

We need to investigate the exponent. By Lemma 4.6, we have that

$$\mathbb{P}(E_{u_n(x)}) = \frac{a_n}{(2n+1)^d} e^{-x}. \tag{4.76}$$

By Proposition 4.11, we have that

$$\lambda_{E_{u_n(x)}} = 1 + o(1). \tag{4.77}$$

Therefore, for any x,

$$\mathbb{P}(\mathcal{M}_n \geq u_n(x)) = 1 - \exp(-a_n e^{-x}) + o(1). \tag{4.78}$$

This completes the proof of Theorem 3.9.

If we further assume that $\mathbb{P}$ has finite supercritical clusters, then by Lemma 4.9 we can take $a_n = 1 + o(1)$. For Theorem 1.2, we note that the further assumption (1.14) implies that $\mathbb{P}$ has finite supercritical clusters, and that with $u_n(x) = u_n + x u_n^{1/d}$, where $u_n = u_n(0)$. Hence, we obtain that

$$\mathbb{P}(E_{u_n(x)}) = \mathbb{P}(E_{u_n}) \frac{\mathbb{P}(E_{u_n + x u_n^{1/d}})}{\mathbb{P}(E_{u_n})} = \mathbb{P}(E_{u_n}) e^{-x\eta^{\frac{d-1}{d}}}[1 + o(1)] = n^{-d} e^{-x\eta^{\frac{d-1}{d}}}[1 + o(1)]. \tag{4.79}$$

The conclusion then follows from (4.78). □

4.4 Proof of Theorems 1.3, 1.4 and 1.5

Proof of Theorems 1.3 and 1.4. We will prove Theorems 1.3 and 1.4 simultaneously. In order to do so, we let $\delta = 1$ for $p < p_c$ and $\delta = \frac{d-1}{d}$ for $p > p_c$. We then assume that

$$-\lim_{n\to\infty} \frac{1}{n^\delta} \log \mathbb{P}(|\mathcal{C}(0)| \geq n) = \xi \tag{4.80}$$

exists. The main ingredient is the following lemma:

Lemma 4.13 (Convergence in probability). *For any $\varepsilon > 0$, there exists $\kappa > 0$ such that as $n \to \infty$,*

$$\mathbb{P}\Big(\Big|\frac{\mathcal{M}_n}{(\log n)^{1/\delta}} - C\Big| > \varepsilon\Big) \leq n^{-\kappa}, \tag{4.81}$$

where $C = d\zeta$ for $p < p_c$ and $C = d^{\frac{d-1}{d}}\eta$ for $p > p_c$.

Before proving Lemma 4.13, we will complete the proofs of Theorems 1.3 and 1.4 subject to Lemma 4.13.

Take $n_k = 2^k$. As a consequence of Lemma 4.13, and the fact that for every $\kappa > 0$,

$$n_k^{-\kappa} = 2^{-\kappa k}$$

is summable in k, we obtain that $\frac{\mathcal{M}_{n_k}}{(\log(n_k))^{1/\delta}}$ converges to C a.s. Thus, we have a.s. convergence along the subsequence $(n_k)_{k\geq 0}$. Moreover, we have that a.s. $n \mapsto \mathcal{M}_n$ is non-decreasing. Therefore, for any $n_k < n \leq n_{k+1}$ we can bound

$$\frac{\mathcal{M}_{n_k}}{(\log(n_k))^{1/\delta}}\Big(\frac{\log(n_k)}{\log(n_{k+1})}\Big)^{1/\delta} \leq \frac{\mathcal{M}_n}{(\log n)^{1/\delta}} \leq \frac{\mathcal{M}_{n_{k+1}}}{(\log(n_{k+1}))^{1/\delta}}\Big(\frac{\log(n_{k+1})}{\log(n_k)}\Big)^{1/\delta}. \tag{4.82}$$

As $n \to \infty$, also $n_k, n_{k+1} \to \infty$. Thus, $\frac{\mathcal{M}_{n_k}}{(\log(n_k))^{1/\delta}}$ and $\frac{\mathcal{M}_{n_{k+1}}}{(\log(n_{k+1}))^{1/\delta}}$ converge a.s. to C. Furthermore,

$$\lim_{k\to\infty} \frac{\log(n_{k+1})}{\log(n_k)} = \lim_{k\to\infty} \frac{k+1}{k} = 1, \tag{4.83}$$

so that both upper and lower bound in (4.82) converge to C almost surely. This completes the proofs of Theorems 1.3 and 1.4. □

Proof of Lemma 4.13. Fix $\varepsilon > 0$. We will prove

$$\mathbb{P}\Big(\frac{\mathcal{M}_n}{(\log n)^{1/\delta}} > C + \varepsilon\Big) \leq n^{-\kappa}, \tag{4.84}$$

and

$$\mathbb{P}\Big(\frac{\mathcal{M}_n}{(\log n)^{1/\delta}} < C - \varepsilon\Big) \leq n^{-\kappa}. \tag{4.85}$$

Let C be the constant such that

$$\mathbb{P}\Big(\frac{|\mathcal{C}(0)|}{(\log n)^{1/\delta}} > C\Big) = n^{-d(1+o(1))}, \tag{4.86}$$

This constant exists by (1.7) in the case $p < p_c$, and by assumption (1.13) (proved in $d = 2, 3$) for $p > p_c$.

With this choice of C, for $\varepsilon > 0$, there exists a $\kappa' \in (0, d)$ such that

$$\mathbb{P}\Big(\frac{|\mathcal{C}(0)|}{(\log n)^{1/\delta}} > C + \varepsilon\Big) \leq n^{-d-\kappa'}, \tag{4.87}$$

while

$$\mathbb{P}\Big(\frac{|\mathcal{C}(0)|}{(\log n)^{1/\delta}} < C - \varepsilon\Big) \leq 1 - n^{-d+\kappa'}. \tag{4.88}$$

To prove (4.84), we use that

$$\begin{aligned}
\mathbb{P}\Big(\frac{\mathcal{M}_n}{(\log n)^{1/\delta}} > C + \varepsilon\Big) &= \mathbb{P}\Big(\bigcup_{x\in B_n} \{|\mathcal{C}(x)| > (C+\varepsilon)(\log n)^{1/\delta}\}\Big) \\
&\leq \sum_{x\in B_n} \mathbb{P}\Big(|\mathcal{C}(0)| > (C+\varepsilon)(\log n)^{1/\delta}\Big) \\
&\leq |B_n| n^{-d-\kappa'} \leq n^{-\kappa},
\end{aligned} \tag{4.89}$$

where we use (4.87).

To prove (4.85), we use that the events $\{|\mathcal{C}(x)| \leq (C+\varepsilon)(\log n)^{1/\delta}\}_{x\in A_n}$ are independent when

$$A_n = (K_n\mathbb{Z})^d \cap B_n. \tag{4.90}$$

and

$$K_n = \lceil (C+\varepsilon)(\log n)^{1/\delta} \rceil \tag{4.91}$$

Therefore,

$$\begin{aligned}
\mathbb{P}\Big(\frac{\mathcal{M}_n}{(\log n)^{1/\delta}} < C-\varepsilon\Big) &= \mathbb{P}\Big(\bigcap_{x\in A_n} \{|\mathcal{C}(x)| \le (C-\varepsilon)(\log n)^{1/\delta}\}\Big)\\
&= \prod_{x\in A_n} \mathbb{P}\Big(|\mathcal{C}(x)| < (C-\varepsilon)(\log n)^{1/\delta}\Big)\\
&\le \mathbb{P}\Big(|\mathcal{C}(0)| < (C-\varepsilon)(\log n)^{1/\delta}\Big)^{|A_n|}. \qquad (4.92)
\end{aligned}$$

We next use (4.88) and the fact that

$$|A_n| \ge \Big(\frac{n}{\lceil (C+\varepsilon)(\log n)^{1/\delta}\rceil}\Big)^d, \tag{4.93}$$

so arrive at a bound, for every $\kappa \in (0,\kappa')$,

$$\mathbb{P}\Big(\frac{\mathcal{M}_n}{(\log n)^{1/\delta}} < C-\varepsilon\Big) \le \big(1-n^{-d+\kappa'}\big)^{|A_n|} \le n^{-\kappa}, \tag{4.94}$$

which completes the proof. □

Proof of Theorem 1.5. We again use (4.86) together with the observation that the events $\{\mathcal{M}_n^{(\mathrm{zb})} \neq \mathcal{M}_n^{(\mathrm{fb})}\}$ and $\{\mathcal{M}_n^{(\mathrm{zb})} \neq \mathcal{M}_n^{(\mathrm{pb})}\}$ are contained in the event that there exists a cluster on the boundary (either with free or periodic boundary conditions) such that there exists an $x \in \partial B_n$ such that $|\mathcal{C}(x)| \ge \mathcal{M}_n^{(\mathrm{zb})}$. By Theorems 1.3 and 1.4, we have that $\mathcal{M}_n^{(\mathrm{zb})} \ge (C-\varepsilon)(\log n)^{1/\delta}$ a.s. By (4.86) and when $\varepsilon > 0$ is sufficiently small, this probability is thus bounded above by

$$n^{d-1}\mathbb{P}\Big(|\mathcal{C}(x)| \ge (C-\varepsilon)(\log n)^{1/\delta}\Big) \le n^{-\kappa}$$

for some $\kappa > 0$. □

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